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Analytical and numerical solutions of radially symmetric inward solidification problems in spherical geometry

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Abstract—A short-time analytical solution is constructed by using a new technique which assumes fictitious initial temperatures in some fictitious extensions of the actual regions. Later, this short-time solution is compared with the numerical solution obtained by the finite difference scheme in which the space grid points change with the freezing front position. Even a small error in the initial values of the solid temperature and freezing front position, which are required for starting the numerical scheme, can, for a short time, give rise to considerable error in the freezing front position. However, the analytical and numerical solutions were found to be in close agreement if the numerical scheme is started with the analytical values of the solid temperature and freezing front.

1. INTRODUCTION

SOLIDIFICATION problems belong to a more general class of problems, commonly known as moving boundary problems, and considerable information on these problems is available in refs. [1–4]. The melting problem is mathematically analogous to the solidification problem.

Exact similarity solution by Neumann [5] has recently motivated some more exact solutions [6, 7] but this approach is not of much help in the present problem. Tao [8, 9] has obtained some analytical solutions pertaining to one-dimensional solidification in a semi-infinite mold. The method of solution requires the calculations of the derivatives of a function of a function and although, in principle, analytical expressions of these derivatives can be written down, in practice only a few coefficients can be determined so that only short-time solutions can in general be obtained. Approximate solutions [10] and perturbation solutions [11–14] are two other important classes of solutions which have been attempted mostly for one-dimensional solidification problems. In the one-dimensional spherical solidification problems studied in refs. [11, 12], the melt is considered at the melting temperature. At the fixed boundary, a constant temperature is prescribed and the perturbation solutions are valid only for small Stefan numbers. It has been observed by Schulze *et al.* [15] that approximate solutions are generally not accurate for short times.

In ref. [17], Boley developed an embedding technique and studied short-time analytical solutions of solidification problems. Grimado and Boley [18] obtained a short-time solution of the ablation of a sphere while Lee and Boley [19] studied the melting of an infinite solid with a spherical cavity for boundary conditions of the first kind. Many more applications of Boley's technique can be found in ref. [20] and its comparison with the present technique can be found in a recent work by Gupta [21].

Although the results are presented here for boundary conditions of the second kind, the present method of solution is valid for boundary conditions of all three kinds [22] which could be time dependent. The melt could be superheated and the physical parameters may have any value. With some minor changes the results for inward spherical solidification can be used for outward solidification and vice versa. The short-time solution of the heat conduction problem can also be studied by the present method. For some parameter values, a considerable solidified thickness can be obtained as shown in Fig. 1.

For numerical work, the Murray and Landis scheme [23] has been chosen because of the simplicity of its execution. Besides, this scheme lends itself to examining the effect of accuracy of the initial values which are required for starting the numerical scheme.

2. PROBLEM FORMULATION

Consider a superheated melt contained in a spherical mold occupying the region $0 \leqslant R \leqslant 1$. The temperature of this melt at time $t = 0$ is a known quantity $\hat{\theta}_L^{(1)}(R)$. The mold is cooled by prescribing known flux $\hat{Q}(t)$ on the surface $R = 1$ of the sphere. If at $t = 0$, $\hat{\theta}_L^{(1)}(1) \neq T_m$ then the time t_m, $t_m > 0$, at which the surface $R = 1$ will attain the temperature T_m can be calculated by solving a pure heat conduction problem as in ref. [24]. Without any loss of generality it can be assumed that the temperature of the melt at $t = t_m$ is a known quantity $\hat{\theta}_L^{(1)}(R)$ such that $\hat{\theta}_L^{(1)}(1) = T_m$. Once

NOMENCLATURE

a	dimensionless constant, kt_m/R_0^2
A_n	coefficients in equation (22)
c	specific heat [J kg^{-1}°C^{-1}]
erf( )	error function
erfc( )	complimentary error function
H_1, H_2, H_3	terms defined by equations (13), (18) and (19), respectively
k	thermal diffusivity [m^2 s^{-1}]
K	thermal conductivity [J m^{-1} °C^{-1} s^{-1}]
l	latent heat of fusion [J kg^{-1}]
p	dummy variable of integration in equations (11), (12), and (14)–(17)
$\hat{Q}(t)$	prescribed flux at $t = 0$ [J m^{-2} s^{-1}]
$Q(V)$	dimensionless flux in equation (6), flux $\cdot R_0/K_S T_m$
Q_n	coefficients in equation (23)
r	radial coordinate [m]
R	dimensionless radius, r/R_0
R_0	radius of the sphere [m]
t	time [s]
t_m	time at which solidification starts at $R = 1$ [s]
T	dimensionless temperature, temperature/T_m
T_m	melting temperature [°C]
V	dimensionless time, $2(a_S y)^{1/2}$
$\hat{X}(y)$	dimensionless freezing front, dimensionless distance from the origin to the freezing front
$X(V)$	dimensionless freezing front, $\hat{X}(y)$
y	dimensionless time, $(t-t_m)/t_m$.

Greek symbols

α	dimensionless constant, $(k_S/k_L)^{1/2}$
β	dimensionless constant, K_L/K_S
$\hat{\theta}_L^{(1)}(R)$	melt temperature at $t = 0$ [°C]
$\theta_L^{(1)}(R)$	dimensionless temperature in equation (2)
$\theta_L^{(2)}(R)$	dimensionless temperature in equation (11)
$\theta_S^{(1)}, \theta_S^{(2)}$	dimensionless temperatures in equation (12)
λ	dimensionless constant, $l/(c_S T_m)$
ρ	density [kg m^{-3}].

Subscripts

L	liquid
S	solid.

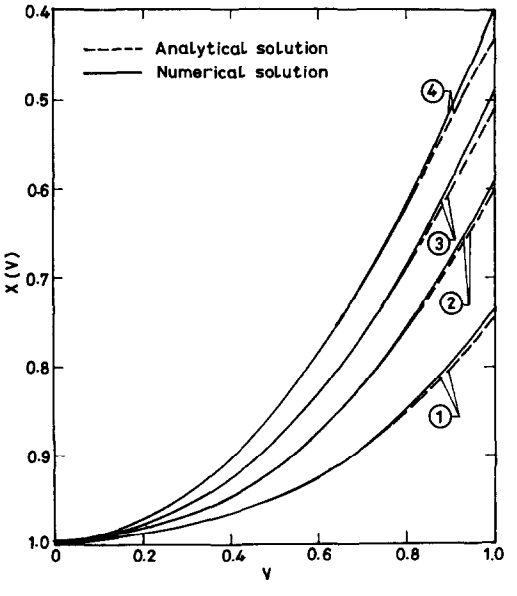

FIG. 1. Freezing front $X(V)$ for copper vs V for different fluxes. $\theta_L^{(1)}(R) = 1.2-0.2R^2$; $a_S = 0.3$, $\lambda = 0.423$, $\beta = 0.503$ and $\alpha = 1.50$. For graphs 1, 2, 3 and 4, $Q(V) = -0.5, -0.75, -1.0$ and -1.25, respectively.

the surface temperature becomes T_m and the cooling continues, the solid–liquid interface will progress towards the interior till the whole of the melt is solidified. The emphasis in this paper is on finding the growth of the freezing front and temperature profiles in the solid and liquid regions.

The following dimensionless differential equations, initial conditions, boundary and interface conditions are to be satisfied.

Liquid region

$$2\alpha^2\left[\frac{\partial T_L}{\partial V} + (1-\rho_s/\rho_L)\frac{dX}{dV}\frac{\partial T_L}{\partial R}\right] = \frac{V}{R}\frac{\partial^2(RT_L)}{\partial R^2}, \quad 0 \leqslant R < X(V), \quad V > 0 \tag{1}$$

$$T_L(R, V)\Big|_{V=0} = \theta_L^{(1)}(R) \tag{2}$$

$$\theta_L^{(1)}(R)\Big|_{R=1} = 1 \tag{3}$$

$$\frac{\partial T_L}{\partial R}\Big|_{R=0} = 0. \tag{4}$$

Solid region

$$2\frac{\partial T_S}{\partial V} = \frac{V}{R}\frac{\partial^2(RT_S)}{\partial R^2}, \quad X(V) < R \leqslant 1, \quad V > 0 \tag{5}$$

$$\frac{\partial T_S}{\partial R}\Big|_{R=1} = Q(V). \tag{6}$$

Solid–liquid interface conditions

$$T_L(R, V)\Big|_{R=X(V)} = 1 \tag{7}$$

$$T_S(R, V)\Big|_{R=X(V)} = 1 \tag{8}$$

$$\left\{\frac{\partial T_S}{\partial R} - \beta\frac{\partial T_L}{\partial R}\right\}\Big|_{R=X(V)} = \frac{2\lambda}{V}\frac{\partial X}{\partial V} \tag{9}$$

$$X(V)\Big|_{V=0} = 1. \tag{10}$$

In the above formulation, the melting temperature T_m is unique. Thermal properties are taken to be constants in any one phase but different phases can have different thermal properties. In equation (1) the densities of solid and liquid are different and so there is natural convection in the liquid. For the present work it will be assumed that the difference in the densities of the two phases is insignificant and so the convective term in equation (1) will be dropped in what follows.

3. SOLUTION

The solution of equation (1) with $\rho_S = \rho_L$ and the solution of equation (5) can be written as

$$T_L(R, V) = \frac{\alpha}{\pi^{1/2}RV}\left[\int_0^1 pH_1(R,p,V)\theta_L^{(1)}(p)\,dp + \int_1^\infty pH_1(R,p,V)\theta_L^{(2)}(p)\,dp\right], \quad 0 \leqslant R < \infty, \quad V > 0 \tag{11}$$

$$T_S(R, V) = \frac{1}{\pi^{1/2}RV}\left[\int_0^1 p\exp\{-(R-p)^2/V^2\} \times \theta_S^{(1)}(p)\,dp + \int_1^\infty p\exp\{-(R-p)^2/V^2\} \times \theta_S^{(2)}(p)\,dp\right], \quad 0 \leqslant R < \infty, \quad V > 0 \tag{12}$$

$$H_1(R,p,V) = \exp\{-\alpha^2(R-p)^2/V^2\} - \exp\{-\alpha^2(R+p)^2/V^2\}. \tag{13}$$

It can be easily checked that T_L satisfies equations (1), (2) and (4) and T_S satisfies equation (5). $\theta_L^{(1)}(R)$ is known and $\theta_L^{(2)}(R)$ is the unknown initial temperature in the fictitious extension $1 \leqslant R < \infty$ of the actual melt region. $\theta_S^{(1)}(R)$ and $\theta_S^{(2)}(R)$ are unknown fictitious initial temperatures in the solid regions $0 \leqslant R < 1$ and $1 < R < \infty$, respectively. Mathematically there are four unknowns, namely, $\theta_L^{(2)}(R)$, $\theta_S^{(1)}(R)$, $\theta_S^{(2)}(R)$ and $X(V)$ and four conditions (6)–(9) to be satisfied. On substituting equations (11) and (12) in conditions (6)–(9) and making some suitable substitutions, the following equations are obtained:

$$\int_{1/V}^{0} p(1-Vp)\exp(-p^2)\theta_S^{(1)}(1-Vp)\,dp + \int_0^{-\infty} p(1-Vp)\exp(-p^2)\theta_S^{(2)}(1-Vp)\,dp = -\frac{\pi^{1/2}}{2}VQ(V) \tag{14}$$

$$\int_{\alpha X(V)}^{\alpha(X-1)/V} H_2(X,p)\theta_L^{(1)}(X-Vp/\alpha)\,dp - \int_{\alpha X(V)}^{\alpha(1+X)/V} H_2(X,p)\theta_L^{(1)}(Vp/\alpha-X)\,dp + \int_{\alpha(X-1)/V}^{-\infty} H_2(X,p)\theta_L^{(2)}(X-Vp/\alpha)\,dp - \int_{\alpha X/V}^{\infty} H_2(X,p)\theta_L^{(2)}(Vp/\alpha-X)\,dp = -\pi^{1/2}X(V) \tag{15}$$

$$\int_{X/V}^{(X-1)/V} H_3(X,p)\theta_S^{(1)}(X-Vp)\,dp + \int_{(X-1)/V}^{-\infty} H_3(X,p)\theta_S^{(2)}(X-Vp)\,dp = -\pi^{1/2}X(V) \tag{16}$$

$$\int_{X/V}^{(X-1)/V} pH_3(X,p)\theta_S^{(1)}(X-Vp)\,dp + \int_{(X-1)/V}^{-\infty} H_3(X,p)\theta_S^{(2)}(X-Vp)\,dp - \beta\alpha\left[\int_{X\alpha/V}^{\alpha(X-1)/V} pH_2(X,p)\theta_L^{(1)}(X-Vp/\alpha)\,dp - \int_{\alpha X/V}^{\alpha(1+X)/V} pH_2(X,p)\theta_L^{(1)}(-X+Vp/\alpha)\,dp + \int_{\alpha(X-1)/V}^{-\infty} pH_2(X,p)\theta_L^2(X-Vp/\alpha)\,dp - \int_{\alpha(X-1)/V}^{\infty} pH_2(X,p)\theta_L^{(2)}(-X+Vp/\alpha)\,dp\right] = \pi^{1/2}V(1-\beta)/2 + \pi^{1/2}\lambda X\frac{dX}{dV} \tag{17}$$

$$H_2(X,p) = (X-Vp/\alpha)\exp(-p^2) \tag{18}$$

$$H_3(X,p) = (X-Vp)\exp(-p^2). \tag{19}$$

The following series expansions for the known and unknown functions will be assumed:

$$\theta_i^{(1)}(R) = \sum_{n=0}^{\infty}\frac{(R-1)^n}{\underline{n}}\frac{\partial^n\theta_i^{(1)}}{\partial R^n}\Big|_{R=1}, \quad 0 \leqslant R \leqslant 1; \quad i = L, S \tag{20}$$

$$\theta_i^{(2)}(R) = \sum_{n=0}^{\infty}\frac{(R-1)^n}{\underline{n}}\frac{\partial^n\theta_i^{(2)}}{\partial R^n}\Big|_{R=1}, \quad 1 \leqslant R < \infty; \quad i = L, S \tag{21}$$

$$X(V) = 1 + \sum_{n=1}^{\infty} A_n V^n, \quad V \geqslant 0 \tag{22}$$

$$Q(V) = \sum_{n=0}^{\infty} Q_n V^n, \quad V \geqslant 0. \tag{23}$$

In order to obtain the solution, series expansions (20)–(23) are substituted in equations (14)–(17) and limits $V \to 0+$ of these equations are taken. Four equations in four unknowns are obtained which when solved give a unique solution. Equations (14)–(17) are then differentiated once with respect to V and limits $V \to 0+$ of the differentiated equations are taken. Once again four equations in four unknowns are obtained which when solved give a unique solution. This process of higher order differentiations and limits $V \to 0+$ can be continued further. Some of the coefficients of the moving boundary are given below:

$$A_1 = 0 \tag{24}$$

$$\theta_L^{(1)}(1) = 1, \quad \theta_L^{(2)}(1) = 1 \tag{25}$$

$$\theta_S^{(1)}(1) = 1, \quad \theta_S^{(2)}(1) = 1 \tag{26}$$

$$4\lambda A_2 = \left.\frac{\partial \theta_S^{(1)}}{\partial R}\right|_{R=1} - \beta \left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1} \tag{27}$$

$$\left.\frac{\partial \theta_S^{(1)}}{\partial R}\right|_{R=1} = \left.\frac{\partial \theta_S^{(2)}}{\partial R}\right|_{R=1} = Q_0 \tag{28}$$

$$\left.\frac{\partial \theta_L^{(2)}}{\partial R}\right|_{R=1} = \left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1} \tag{29}$$

$$6\pi^{1/2}\lambda\alpha A_3 = \pi^{1/2}\alpha Q_1 + \beta \left.\frac{\partial^2 \theta_L^{(1)}}{\partial R^2}\right|_{R=1} + 2\beta(1+2A_2\alpha^2) \left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1} \tag{30}$$

$$\left.\frac{\partial^2 \theta_L^{(2)}}{\partial R^2}\right|_{R=1} = -\left.\frac{\partial^2 \theta_L^{(1)}}{\partial R^2}\right|_{R=1} - 4(1+2A_2\alpha^2) \left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1} \tag{31}$$

$$\left.\frac{\partial^2 \theta_S^{(2)}}{\partial R^2}\right|_{R=1} - \left.\frac{\partial^2 \theta_S^{(1)}}{\partial R^2}\right|_{R=1} = 2\pi^{1/2} Q_1 \tag{32}$$

$$\left.\frac{\partial^2 \theta_S^{(2)}}{\partial R^2}\right|_{R=1} + \left.\frac{\partial^2 \theta_S^{(1)}}{\partial R^2}\right|_{R=1} = -4Q_0(1+2A_2). \tag{33}$$

The coefficients A_4 and A_5 were also calculated for the numerical work. Although, in principle, other coefficients, A_6, A_7, etc. can also be determined, the algebra goes on becoming lengthier. Along with the unknowns of the moving boundary, the unknowns in the temperature solutions are also determined. The temperature $T_L(R, V)$ in the melt is given below. Series expansions (20) and (21) are substituted in equations (11) and (12), respectively, and term by term integration is done. For small values of V and $|R-1|$ the temperature $T_L(R, V)$ can be obtained as follows:

$$\begin{aligned} T_L(R, V) = {} & \frac{1}{2}\operatorname{erfc}\{\alpha(R-1)/V\} \\ & \times \left\{1 + (R-1)\left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1} + \frac{(R-1)^2}{2}\left.\frac{\partial^2 \theta_L^{(1)}}{\partial R^2}\right|_{R=1}\right\} \\ & - \frac{V}{(2\pi^{1/2}R\alpha)}\left\{1 + (2R-1)\left.\frac{\partial \theta_L^{(1)}}{\partial R}\right|_{R=1}\right. \\ & \left. + R(R-1)\left.\frac{\partial^2 \theta_L^{(1)}}{\partial R^2}\right|_{R=1}\right\}\exp\{-\alpha^2(R-1)^2/V^2\} \\ & + \tfrac{1}{2}[2 - \operatorname{erfc}\{\alpha(R-1)/V\}] \\ & \times \left\{1 + (R-1)\left.\frac{\partial \theta_L^{(2)}}{\partial R}\right|_{R=1} + \frac{(R-1)^2}{2}\left.\frac{\partial^2 \theta_L^{(2)}}{\partial R^2}\right|_{R=1}\right\} \\ & + \frac{V}{(2\pi^{1/2}R\alpha)}\left\{1 + (2R-1)\left.\frac{\partial \theta_L^{(2)}}{\partial R}\right|_{R=1}\right. \\ & \left. + R(R-1)\left.\frac{\partial^2 \theta_L^{(2)}}{\partial R^2}\right|_{R=1}\right\}\exp\{-\alpha^2(R-1)^2/V^2\} \\ & + \text{terms of the type } (R-1)^m V^n \\ & \text{where } m+n > 2, R > 0, V > 0. \end{aligned} \tag{34}$$

For numerical work, terms of the type $(R-1)^m B V^n$ where $m+n = 4$ were also calculated in equation (34). In calculating the above temperature wherever the limit of integration is R/V it is taken as ∞ and it is justified as the integrals are error function integrals [25]. Without making this assumption the integrals can also be evaluated but the temperature expression becomes very lengthy. T_L is physically meaningful only for $0 \leqslant R \leqslant X(V)$. In order to obtain the temperature $T_S(R, V)$, put $\alpha = 1$ in equation (34) and replace the subscript L by the subscript S everywhere.

Equations (25), (26) and (28) can be written purely from physical considerations also. For $V \ll 1$, the behaviour of the moving boundary is in agreement with the earlier findings [26] which are valid irrespective of the geometry. The moving boundary and temperature distributions in the Neumann problem [5] can be obtained exactly if a formulation similar to the one presented above with appropriate changes is done. Some of the steps in the method of solution can be rigorously justified by giving the arguments similar to those given in ref. [21].

3.1. *Heat conduction without phase change*

The temperature of the superheated melt occupying the radially symmetric region $0 \leqslant R \leqslant 1$ which is cooled at $R = 1$ is also given by equation (11). The short-time temperature solution in this case is still given by equation (34) provided $\theta_L^{(2)}$ is determined by the prescribed boundary condition.

3.2. *Solidification in an infinite medium with spherical cavity*

The formulation given in equations (1)–(10) can be used for the spherical cavity problem also after

making some changes. For example, the region occupied by the melt at $t = 0$ will now be $1 \leqslant R < \infty$ and equation (4) is not required. T_L for the cavity problem is still given by equation (11) but H_1 in equation (13) will now contain only the first term. $\theta_L^{(2)}$ will now be a known function and $\theta_L^{(1)}$ becomes an unknown function. The freezing front and temperatures can be determined by following the method given earlier.

4. NUMERICAL SOLUTION AND DISCUSSION

The manner in which the Murray and Landis [23] numerical scheme was executed in the present work is being discussed briefly as some discretion can be used in its execution. Thirteen and twenty one space grid points were used in the solid and liquid regions, respectively. At the first time step $\Delta y = 0.001$ in the numerical scheme, the values of the freezing front and the temperatures at the space grid points in the solidified thickness, were taken from the analytical solution and these values will be referred to as starting values in the following. Liquid temperatures at time Δy were calculated using the implicit scheme. At time $2\Delta y$, the freezing front was calculated from the heat balance equation (9) in which the derivatives of the temperatures were calculated at the previous time step using three point formulas incorporating the isotherm conditions. Once the freezing front at time $2\Delta y$ is known, the temperatures T_L and T_S at the new space grid points and time $2\Delta y$ can be calculated using the implicit scheme. For all subsequent time steps, the above procedure was repeated. The analytical solution was used in the numerical scheme only at the first time step. No significant change in the numerical results was observed when the number of spacial grid points in the solid and liquid regions were doubled or the time step was halved.

A large number of numerical experiments were carried out for different parameters and different fluxes, etc. and the following was observed. If for given parameter values, the coefficients in the moving boundary are systematically decreasing in absolute value, then by calculating $|A_n V^n|$ for a given V, where A_n is the last coefficient calculated in the moving boundary, one can easily check whether it makes any significant difference in $\sum_{m=2}^{n} A_m V^m$ or not. If it does not, then the analytical solution is valid at least for this particular value of V. In fact the analytical solution was in general found to be valid for values of V larger than this particular value.

From Table 1, it is clear that for a short time, even a small error in the starting values can make considerable error in the freezing front position. As the solidified thickness increases, this error gradually decreases. In Fig. 1, the solidified thickness for copper is reported. Similar types of results were obtained for other metals such as aluminium. For small values of V and $|R-1|$, there was good agreement between the temperatures calculated from equation (34) and the numerical scheme.

Table 1. Effect of accuracy of the starting values on the freezing front in the numerical solution. Data as in Fig. 1

$Q(V)$	V	Solidified thickness	Percentage error
		0.00196†	0.0
	0.1039	0.00285‡	45.4
		0.00513§	161.7
		0.00760	0.0
	0.202	0.00841	10.52
		0.01063	39.8
−0.5			
		0.01824	0.0
	0.307	0.01901	4.16
		0.02119	16.17
		0.05860	0.0
	0.531	0.05947	1.48
		0.06167	5.2

†Numerical solution started with exact analytical values.
‡Starting solid temperature = analytical temperature − 0.00001 and freezing front = $X(V)+0.00001$.
§Starting solid temperature = analytical temperature − 0.00004 and freezing front = $X(V)+0.00002$.

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SOLUTIONS ANALYTIQUE ET NUMERIQUE D'UN PROBLEME DE SOLIDIFICATION PENETRANTE, A SYMETRIE RADIALE DANS UNE GEOMETRIE SPHERIQUE

Résumé—Une solution analytique rapide est obtenue en utilisant une technique nouvelle qui suppose des températures initiales fictives dans des extensions fictives des régions réelles. Ultérieurement, cette solution est comparée avec la solution numérique obtenue par différences finies dans laquelle les points de la grille spatiale changent avec la position du front de solidification. Une petite erreur dans les valeurs initiales de la température du solide et dans la position du front de solidification, données nécessaires pour initialiser le schéma numérique, peut conduire dans un temps court à une erreur considérable sur la position du front de solidification. Néanmoins les solutions analytique et numérique sont en bon accord si le schéma numérique est initialisé avec les valeurs analytiques de la température du solide et du front de solidification.

ANALYTISCHE UND NUMERISCHE LÖSUNGEN RADIALSYMMETRISCHER ERSTARRUNGSERSCHEINUNGEN IM INNEREN VON KUGELFÖRMIGEN GEOMETRIEN

Zusammenfassung—Man erhält eine analytische Kurzzeitlösung mit Hilfe einer neuen Technik, bei der fiktive Anfangstemperaturen in fiktiven Bereichen der betrachteten Regionen angenommen werden. Die Kurzzeitlösung wird später mit der numerischen Lösung verglichen, die man durch das Finite-Differenzen-Schema erhält, und bei der sich die Raumlage der Gitterpunkte mit der Position der Erstarrungsfront ändert. Schon geringfügige Fehler bei den für den Start des numerischen Verfahrens notwendigen Anfangswerten für die Festkörpertemperatur und Lage der Erstarrungsfront können für kurze Zeiten den Fehler bei der Ermittlung der Lage der Erstarrungsfront erheblich anwachsen lassen. Die analytische und die numerische Lösung stimmen jedoch gut miteinander überein, wenn als Startwerte für das numerische Verfahren die analytischen Werte der Festkörpertemperatur und der Erstarrungsfront verwendet werden.

АНАЛИТИЧЕСКОЕ И ЧИСЛЕННОЕ РЕШЕНИЕ РАДИАЛЬНО-СИММЕТРИЧНЫХ ВНУТРЕННИХ ЗАДАЧ ЗАТВЕРДЕВАНИЯ ДЛЯ СФЕРИЧЕСКИХ ТЕЛ

Аннотация—С помощью нового метода, предполагающего фиктивные температуры на некоторых фиктивных участках реальных областей, построено аналитическое решение для малых времен. Далее проведено сравнение решения с численным решением, полученным с применением конечно-разностной схемы, пространственная сетка которой изменяется с изменением положения фронта затвердевания. При численных расчетах даже небольшие погрешности в начальных значениях температуры и положения фронта затвердевания могут на малых временах привести к значительным ошибкам. Однако сравнение показало, что аналитическое и численное решения хорошо согласуются, если в численных расчетах используются начальные значения в аналитическом виде для температуры твердого тела и фронта затвердевания.